

# Jet Finslerian geometry of the conformal Minkowski metric

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## Abstract

The paper develops the Finsler-like geometry on the 1-jet space for the jet conformal Minkowski (JCM) metric, which naturally extends the Minkowski metric in the Chernov-Pavlov framework. To this aim there are determined the nonlinear connection, distinguished (d-) Cartan linear connection, d-torsions and d-curvatures. The field geometrical gravitational and electromagnetic d-models based on the JCM metric are discussed.

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**Key words and phrases:** metric structure; canonical nonlinear connection; Cartan canonical linear connection; d-torsions; d-curvatures; extended Einstein geometrical equations.

## 1 Introduction

It is obvious that our genuine physical intuition distinguishes four dimensions in a natural correspondence with the material reality. Consequently, the four dimensionality plays a special role in almost all modern physical theories [9, 16, 17].

On the other hand, it is a well known fact that, in order to create Relativity Theory, Einstein used Riemannian geometry instead of classical Euclidean geometry, the first one representing a natural mathematical model for *local isotropic space-time*. Although the use of Riemannian geometry was indeed a genial idea, there are recent studies of physicists that suggest a *non-isotropic* perspective of space-time. For example, in Pavlov's works [15, 16, 17], the concept of inertial body mass emphasizes the necessity to study *local non-isotropic spaces*. For the study of non-isotropic physical phenomena, Finsler geometry proves to be adequate and proficient as mathematical framework.

Recent studies of Russian scholars (e.g., Asanov [1], Garas'ko [7] and Pavlov [8], [15]) emphasize the importance of Finsler geometry, which is characterized by the total equality in rights of all non-isotropic directions. For such a reason, in their works is underlined the important role played in theory of space-time structure and gravitation (as well as in unified gauge field theories) by the  $m$ -root metric ([18, 3, 5])

$$L : TM \rightarrow \mathbb{R}, \quad L(x, y) = \sqrt[m]{a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}}.$$

It is known that the 1-jet fibre bundle is a basic object in the study of classical and quantum field theories (Olver, [14]). For these geometrical and physical reasons, the present paper is devoted to the construction on the 1-jet space  $J^1(\mathbb{R}, M^4)$  of the Finsler-like geometry (together with the extended gravitational and electromagnetic geometrical models) for the *jet conformal Minkowski (JCM) metric*  $F : J^1(\mathbb{R}, M^4) \rightarrow \mathbb{R}$ , defined by<sup>1</sup>

$$F(t, x, y) = e^{\sigma(x)} \cdot \sqrt{h^{11}(t)} \cdot \sqrt{y_1^1 y_1^2 + y_1^1 y_1^3 + y_1^1 y_1^4 + y_1^2 y_1^3 + y_1^2 y_1^4 + y_1^3 y_1^4}, \quad (1.1)$$

where  $\sigma(x)$  is a smooth non-constant function on  $M^4$ ,  $h^{11}(t)$  is the dual of the Riemannian metric  $h_{11}(t)$  on  $\mathbb{R}$  and

$$(t, x, y) = (t, x^1, x^2, x^3, x^4, y_1^1, y_1^2, y_1^3, y_1^4)$$

are the coordinates of the 1-jet space  $J^1(\mathbb{R}, M^4)$ . These transform by the rules:

$$\tilde{t} = \tilde{t}(t), \quad \tilde{x}^p = \tilde{x}^p(x^q), \quad \tilde{y}_1^p = \frac{\partial \tilde{x}^p}{\partial x^q} \frac{dt}{d\tilde{t}} \cdot y_1^q, \quad p, q = \overline{1, 4}, \quad (1.2)$$

where  $d\tilde{t}/dt \neq 0$  and  $\text{rank}(\partial \tilde{x}^p / \partial x^q) = 4$ .

**Remark.** It is easy to verify that (as emphasized in the recent studies [6] and [15]) the geometrical object

$$G_{11}(y) \stackrel{\text{def}}{=} y_1^1 y_1^2 + y_1^1 y_1^3 + y_1^1 y_1^4 + y_1^2 y_1^3 + y_1^2 y_1^4 + y_1^3 y_1^4 \quad (1.3)$$

is a quadratic form in  $y = (y_1^1, y_1^2, y_1^3, y_1^4)$ , whose canonical form is the Minkowski metric. Namely, denoting  $x = (x^1, x^2, x^3, x^4)$ ,  $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^4)$

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<sup>1</sup>In the following we shall reduce the domain of the constructed geometric objects in order to ensure their existence and, where this is required, their smoothness. As well, we shall implicitly use throughout the work the Einstein convention of summation.

and  $A = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{3} & 1 & -1/\sqrt{6} \\ 1/\sqrt{6} & 2/\sqrt{3} & 0 & -1/\sqrt{6} \\ 1/\sqrt{6} & 0 & 0 & 3/\sqrt{6} \\ 1/\sqrt{6} & -1/\sqrt{3} & -1 & -1/\sqrt{6} \end{pmatrix}$ , if we apply on the product manifold  $\mathbb{R} \times M^4$  the invertible linear coordinate transformation

$$t = \tilde{t}, \quad {}^T x = A \cdot {}^T \tilde{x},$$

then in the induced coordinates  $(\tilde{t}, \tilde{x}, \tilde{y})$  on  $J^1(\mathbb{R}, M^4)$ , we have  ${}^T y = A \cdot {}^T \tilde{y}$ , and the JCM Finslerian metric (1.1) has the particular form,

$$F(\tilde{t}, \tilde{x}, \tilde{y}) = e^{\sigma(\tilde{x} \cdot {}^T A)} \cdot \sqrt{h^{11}(\tilde{t})} \cdot \sqrt{(\tilde{y}_1^1)^2 - (\tilde{y}_1^2)^2 - (\tilde{y}_1^3)^2 - (\tilde{y}_1^4)^2}.$$

The distinguished (d-) jet framework ([10, 12, 1, 2]), which involves specific geometric objects as canonical nonlinear connection, Cartan canonical linear connection, d-torsions, d-curvatures, and their related extended gravitational and electromagnetic geometrical models produced by an arbitrary jet Lagrangian function

$$L : J^1(\mathbb{R}, M^n) \rightarrow \mathbb{R},$$

was completely treated in recent works of the authors of this paper ([4, 13]). We point out that geometrical ideas from these works are similar, but distinct, from those promoted by Miron and Anastasiei in the classical Lagrangian geometry on tangent bundle ([10]). Namely, the case of the present framework was initially stated by Asanov in [2], and further generalized in [12] by the second author of this paper.

In the sequel, we apply the general geometrical results from [4] and [13] to the particular jet conformal Minkowski-metric (1.1).

## 2 The canonical nonlinear connection of the model

Let  $(\mathbb{R}, h_{11}(t))$  be a Riemannian manifold, where  $\mathbb{R}$  is the set of real numbers. The Christoffel symbol of the Riemannian metric  $h_{11}(t)$  is

$$\varkappa_{11}^1 = \frac{h^{11}}{2} \frac{dh_{11}}{dt}, \quad \text{where } h^{11} = (h_{11})^{-1} > 0.$$

Let also  $M^4$  be a manifold of dimension four, whose local coordinates are  $x = (x^1, x^2, x^3, x^4)$ . These manifolds produce the 1-jet space  $J^1(\mathbb{R}, M^4)$ , whose local coordinates are  $(t; x; y)$ , where  $y = (y_1^1, y_1^2, y_1^3, y_1^4)$ .

Let's consider on  $J^1(\mathbb{R}, M^4)$  the JCM metric (1.1), whose domain of definition consists of all values  $(t; x; y)$  which satisfy the condition  $G_{11}(y) > 0$ , where  $G_{11}$  is given by (1.3). If we use the notation

$$S_{[1]1} = y_1^1 + y_1^2 + y_1^3 + y_1^4,$$

then the following relations are true:

$$\begin{aligned} G_{i1} &\stackrel{def}{=} \frac{\partial G_{11}}{\partial y_1^i} = S_{[1]1} - y_1^i, \\ G_{ij} &\stackrel{def}{=} \frac{\partial G_{i1}}{\partial y_1^j} = \frac{\partial^2 G_{11}}{\partial y_1^i \partial y_1^j} = 1 - \delta_{ij}, \end{aligned}$$

where  $\delta_{ij}$  is the Kronecker symbol. Obviously, the homogeneity of degree 2 of the "y-function"  $G_{11}$  (which is in fact a d-tensor on  $J^1(\mathbb{R}, M^4)$ ) leads to the equalities:

$$G_{i1}y_1^i = 2G_{11}, \quad G_{ij}y_1^i y_1^j = 2G_{11}.$$

By direct computation, we get

**Lemma 2.1.** *a) The fundamental metrical d-tensor produced by the JCM Finslerian metric  $F$  is given by the formula*

$$g_{ij}(t, x, y) = \frac{h_{11}(t)}{2} \frac{\partial^2 F^2}{\partial y_1^i \partial y_1^j},$$

which in our case leads to

$$g_{ij}(x) = \frac{e^{2\sigma(x)}}{2} (1 - \delta_{ij}), \quad (2.1)$$

and the matrix  $g = (g_{ij})$  admits the inverse  $g^{-1} = (g^{jk})$ , whose entries are

$$g^{jk}(x) = \frac{2e^{-2\sigma(x)}}{3} (1 - 3\delta^{jk}).$$

b) The divergence of the  $\sigma$ -diagonal vector field on  $M^4$

$$D_\sigma = \sigma(x) \frac{\partial}{\partial x^1} + \sigma(x) \frac{\partial}{\partial x^2} + \sigma(x) \frac{\partial}{\partial x^3} + \sigma(x) \frac{\partial}{\partial x^4}.$$

has the expression

$$\operatorname{div} D_\sigma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4,$$

where  $\sigma_i = \partial\sigma/\partial x^i$ .

Hence, using the general results from [13], we yield:

**Proposition 2.2.** *For the conformal JCM metric (1.1), the energy action functional*

$$\begin{aligned}\mathbb{E}(t, x(t)) &= \int_a^b F^2(t, x, y) \sqrt{h_{11}} dt = \int_a^b e^{2\sigma(x)} \cdot G_{11}(y) \cdot h^{11}(t) \sqrt{h_{11}(t)} dt \\ &= \int_a^b e^{2\sigma(x)} (y_1^1 y_1^2 + y_1^1 y_1^3 + y_1^1 y_1^4 + y_1^2 y_1^3 + y_1^2 y_1^4 + y_1^3 y_1^4) h^{11} \sqrt{h_{11}} dt,\end{aligned}$$

where  $y = dx/dt$ , produces on the 1-jet space  $J^1(\mathbb{R}, M^4)$  the **canonical nonlinear connection**

$$\Gamma = \left( M_{(1)1}^{(i)} = -\varkappa_{11}^1 y_1^i, \quad N_{(1)j}^{(i)} \right), \quad (2.2)$$

where

$$N_{(1)j}^{(i)} = \sigma_j y_1^i + \sigma_m y_1^m \delta_j^i + \left[ \sigma_i - \frac{1}{3} \operatorname{div} D_\sigma \right] (S_{[1]1} - y_1^j).$$

*Proof.* The Euler-Lagrange equations of the energy action functional  $\mathbb{E}$  are

$$\frac{d^2 x^i}{dt^2} + 2H_{(1)1}^{(i)}(t, x^k, y_1^k) + 2G_{(1)1}^{(i)}(t, x^k, y_1^k) = 0, \quad y_1^k = \frac{dx^k}{dt}, \quad (2.3)$$

where we have the local geometrical components

$$\left\{ \begin{array}{ll} H_{(1)1}^{(i)} & \stackrel{def}{=} -\frac{1}{2} \varkappa_{11}^1(t) y_1^i \\ G_{(1)1}^{(i)} & \stackrel{def}{=} \frac{h_{11} g^{ik}}{4} \left[ \frac{\partial^2 F^2}{\partial x^m \partial y_1^k} y_1^m - \frac{\partial F^2}{\partial x^k} + \frac{\partial^2 F^2}{\partial t \partial y_1^k} + \right. \\ & \quad \left. + \frac{\partial F^2}{\partial y_1^k} \varkappa_{11}^1(t) + 2h^{11} \varkappa_{11}^1 g_{km} y_1^m \right] = \\ & = \sigma_m y_1^m y_1^i + \left[ \sigma_i - \frac{1}{3} \operatorname{div} D_\sigma \right] \cdot G_{11} \end{array} \right.$$

which determine a *semispray* on the 1-jet space  $J^1(\mathbb{R}, M^4)$ . Its associated *canonical nonlinear connection* has the general form [11, 13])

$$\Gamma = \left( M_{(1)1}^{(i)} = 2H_{(1)1}^{(i)} = -\varkappa_{11}^1 y_1^i, \quad N_{(1)j}^{(i)} = \frac{\partial G_{(1)1}^{(i)}}{\partial y_1^j} \right).$$

□

### 3 Cartan canonical linear connection, d-torsions and d-curvatures

The canonical nonlinear connection (2.2) is essential in constructing the dual *adapted bases* of distinguished (d-) vector fields

$$\left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \varkappa_{11}^1 y_1^p \frac{\partial}{\partial y_1^p} ; \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)i}^{(p)} \frac{\partial}{\partial y_1^p} ; \frac{\partial}{\partial y_1^i} \right\} \subset \mathcal{X}(E) \quad (3.1)$$

and distinguished covector fields

$$\left\{ dt ; dx^i ; \delta y_1^i = dy_1^i - \varkappa_{11}^1 y_1^i dt + N_{(1)p}^{(i)} dx^p \right\} \subset \mathcal{X}^*(E), \quad (3.2)$$

where  $E = J^1(\mathbb{R}, M^4)$ . Note that, under a change of coordinates (1.2), the elements of the adapted bases (3.1) and (3.2) transform as classical tensors. Consequently, all subsequent geometrical objects on the 1-jet space  $J^1(\mathbb{R}, M^4)$ , like Cartan canonical linear connection, torsion, curvature etc., will be described in local adapted components.

In this respect, using a general result from [13], by direct computations, we have the following

**Proposition 3.1.** *The Cartan canonical  $\Gamma$ -linear connection, produced by the jet conformal Minkowski metric (1.1), has the following adapted local components:*

$$C\Gamma = \left( \varkappa_{11}^1, G_{j1}^k = 0, L_{jk}^i, C_{j(k)}^{i(1)} = 0 \right), \quad (3.3)$$

where

$$L_{jk}^i = \delta_j^i \sigma_k + \delta_k^i \sigma_j + (1 - \delta_{jk}) \sigma_i - \frac{1 - \delta_{jk}}{3} \operatorname{div} D_\sigma.$$

*Proof.* Using the local derivative operators (3.1) and the general formulas which provide the adapted components of the Cartan canonical connection ([13]), we get

$$\begin{cases} G_{j1}^k = \frac{g^{km}}{2} \frac{\delta g_{mj}}{\delta t} = 0, & L_{jk}^i = \frac{g^{im}}{2} \left( \frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right), \\ C_{j(k)}^{i(1)} = \frac{g^{im}}{2} \left( \frac{\partial g_{jm}}{\partial y_1^k} + \frac{\partial g_{km}}{\partial y_1^j} - \frac{\partial g_{jk}}{\partial y_1^m} \right) = 0. \end{cases}$$

□

**Remark.** It is straightforward to check the relation  $L_{jk}^i = \frac{\partial N_{(1)j}^{(i)}}{\partial y_1^k}$ , which, considering the homogeneity of degree 1 of the local functions  $N_{(1)j}^{(i)}$ , leads to

$$\frac{\partial N_{(1)j}^{(i)}}{\partial y_1^m} y_1^m = N_{(1)j}^{(i)} \Leftrightarrow L_{jm}^i y_1^m = N_{(1)j}^{(i)}. \quad (3.4)$$

**Proposition 3.2.** *The Cartan canonical  $\Gamma$ -linear connection  $C\Gamma$  of the jet conformal Minkowski metric (1.1) has a **single** effective local torsion d-tensor, namely*

$$R_{(1)jk}^{(l)} = \mathfrak{R}_{pjk}^l y_1^p,$$

where, using the notations

$$\sigma_{ij} = \frac{\partial^2 \sigma}{\partial x^i \partial x^j}, \quad \text{grad } \sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4), \quad ||\text{grad } \sigma||^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2,$$

$$(\text{div } D\sigma)_i = \frac{\partial(\text{div } D\sigma)}{\partial x^i} = \sigma_{1i} + \sigma_{2i} + \sigma_{3i} + \sigma_{4i}$$

we have

$$\begin{aligned} \mathfrak{R}_{ijk}^l = & \delta_j^l (\sigma_{ik} - \sigma_i \sigma_k) - \delta_k^l (\sigma_{ij} - \sigma_i \sigma_j) + \\ & + (1 - \delta_{ij}) (\sigma_{lk} - \sigma_l \sigma_k) - (1 - \delta_{ik}) (\sigma_{lj} - \sigma_l \sigma_j) + \\ & + \frac{1}{3} (\text{div } D\sigma) (\sigma_k - \sigma_j + \delta_{ik} \sigma_j - \delta_{ij} \sigma_k) + \\ & + \left[ ||\text{grad } \sigma||^2 - \frac{1}{3} (\text{div } D\sigma)^2 \right] (\delta_k^l - \delta_j^l + \delta_{ik} \delta_j^l - \delta_{ij} \delta_k^l) + \\ & + \frac{1}{3} [(\text{div } D\sigma)_j - (\text{div } D\sigma)_k + \delta_{ij} (\text{div } D\sigma)_k - \delta_{ik} (\text{div } D\sigma)_j]. \end{aligned} \quad (3.5)$$

*Proof.* A general  $h$ -normal  $\Gamma$ -linear connection on the 1-jet space  $J^1(\mathbb{R}, M^4)$  is characterized by *eight* effective d-tensors of torsion ([13]). For our Cartan canonical connection (3.3), these reduce only to *one* (the other seven cancel):

$$R_{(1)jk}^{(l)} = \frac{\delta N_{(1)j}^{(l)}}{\delta x^k} - \frac{\delta N_{(1)k}^{(l)}}{\delta x^j}.$$

Using now the expressions of the derivatives  $\delta/\delta x^i$ , formula (3.4) and the  $y$ -independence  $L_{jk}^i = L_{jk}^i(x)$ , we find

$$R_{(1)jk}^{(l)} = \mathfrak{R}_{pjk}^l y_1^p,$$

where

$$\mathfrak{R}_{ijk}^l := \frac{\partial L_{ij}^l}{\partial x^k} - \frac{\partial L_{ik}^l}{\partial x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l.$$

Finally, laborious computations lead to the expression (3.5) of the d-tensor  $\mathfrak{R}_{ijk}^l$ .  $\square$

**Proposition 3.3.** *The Cartan canonical  $\Gamma$ -linear connection  $CT$  of the jet conformal Minkowski metric (1.1) has a **single** effective local curvature d-tensor, namely*

$$R_{ijk}^l = \mathfrak{R}_{ijk}^l,$$

where  $\mathfrak{R}_{ijk}^l$  is given by (3.5).

*Proof.* A general  $h$ -normal  $\Gamma$ -linear connection on the 1-jet space  $J^1(\mathbb{R}, M^4)$  is characterized by *five* effective d-tensors of curvature ([13]). For our Cartan canonical connection (3.3), these reduce only to *one* (the other four cancel), namely

$$\begin{aligned} R_{ijk}^l &\stackrel{\text{def}}{=} \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l + C_{i(r)}^{l(1)} R_{(1)jk}^{(r)} = \\ &= \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l = \\ &= \frac{\partial L_{ij}^l}{\partial x^k} - \frac{\partial L_{ik}^l}{\partial x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l = \mathfrak{R}_{ijk}^l. \end{aligned}$$

$\square$

## 4 Geometrical field model produced by the jet conformal Minkowski metric

### 4.1 Gravitational-like geometrical model

From a geometric-physical point of view, on the 1-jet space  $J^1(\mathbb{R}, M^4)$ , the jet conformal Minkowski metric (1.1) produces the adapted metrical d-tensor

$$\mathbb{G} = h_{11} dt \otimes dt + g_{ij} dx^i \otimes dx^j + h^{11} g_{ij} \delta y_1^i \otimes \delta y_1^j, \quad (4.1)$$

where  $g_{ij}$  is given by (2.1). This may be regarded as a “*non-isotropic gravitational potential*”. In such a “physical” context, the nonlinear connection



$\Gamma$  (used in the construction of the distinguished 1-forms  $\delta y_1^i$ ) prescribes, probably, a kind of “*interaction*” between  $(t)$ -,  $(x)$ - and  $(y)$ -fields.

We postulate that the non-isotropic gravitational potential  $\mathbb{G}$  is governed by the *geometrical Einstein equations*

$$\text{Ric } (CT) - \frac{\text{Sc } (CT)}{2} \mathbb{G} = \mathcal{K} \mathcal{T}, \quad (4.2)$$

where  $\text{Ric } (CT)$  is the *Ricci d-tensor* associated to the Cartan canonical connection  $CT$  (in Riemannian sense and using adapted bases),  $\text{Sc } (CT)$  is the *scalar curvature*,  $\mathcal{K}$  is the *Einstein constant* and  $\mathcal{T}$  is the intrinsic *stress-energy* d-tensor of matter ([10, 4]).

In this way, working with the adapted basis of vector fields (3.1), we can find the local geometrical Einstein equations for the metric (1.1). Firstly, by direct computations, we find:

**Lemma 4.1.** *The Ricci d-tensor of the Cartan canonical connection  $CT$  of the metric (1.1) has a **single** effective local Ricci d-tensor, namely*

$$\begin{aligned} R_{ij} = & -2(\sigma_{ij} - \sigma_i \sigma_j) + \\ & + \frac{1 - \delta_{ij}}{3} \left[ 3\Delta\sigma + 6 \|\text{grad } \sigma\|^2 - 2(\text{div } D_\sigma)^2 - \mathfrak{S} \right], \end{aligned} \quad (4.3)$$

where

$$\Delta\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33} + \sigma_{44}, \quad \mathfrak{S} = \sum_{p,q=1}^4 \sigma_{pq}.$$

*Proof.* A general  $h$ -normal  $\Gamma$ -linear connection on the 1-jet space  $J^1(\mathbb{R}, M^4)$  is characterized by *six* effective Ricci d-tensors ([13]). For our Cartan canonical connection (3.3), these reduce only to *one* (the other five cancel):

$$R_{ij} \stackrel{\text{def}}{=} R_{ijm}^m = \mathfrak{R}_{ijm}^m.$$

Then, a direct computation gives the expression (4.3) of the Ricci d-tensor  $R_{ij}$ .  $\square$

**Lemma 4.2.** *The scalar curvature  $\text{Sc } (CT)$  of the Cartan canonical connection  $CT$  of the jet conformal Minkowski metric (1.1) is given by*

$$R = 4e^{-2\sigma} \left[ 3\Delta\sigma + 3 \|\text{grad } \sigma\|^2 - (\text{div } D_\sigma)^2 - \mathfrak{S} \right]. \quad (4.4)$$

*Proof.* The general formula for the scalar curvature of the Cartan connection reduces to ([13])

$$\text{Sc } (C\Gamma) \stackrel{\text{def}}{=} g^{pq} R_{pq} := R,$$

where  $R$  is given by (4.4).  $\square$

By describing the global geometrical Einstein equations (4.2) in the adapted basis of vector fields (3.1), we find the following important geometrical and physical result ([13]):

**Proposition 4.3.** *The local **geometrical Einstein equations** that govern the non-isotropic gravitational potential  $\mathbb{G}$  (produced by the jet conformal Minkowski metric (1.1)) are given by:*

$$R_{ij} - \frac{R}{2} g_{ij} = \mathcal{K} \mathcal{T}_{ij}, \quad (4.5)$$

$$\begin{cases} -Rh_{11} = 2\mathcal{K}\mathcal{T}_{11}, & 0 = \mathcal{T}_{1i}, & 0 = \mathcal{T}_{i1}, & 0 = \mathcal{T}_{(i)1}^{(1)}, \\ 0 = \mathcal{T}_{1(i)}^{(1)}, & 0 = \mathcal{T}_{i(j)}^{(1)}, & 0 = \mathcal{T}_{(i)j}^{(1)}, & -Rh^{11}g_{ij} = 2\mathcal{K}\mathcal{T}_{(i)(j)}^{(1)(1)}. \end{cases} \quad (4.6)$$

**Remark.** The Einstein geometrical equations (4.5) and (4.6) impose that the stress-energy d-tensor of matter  $\mathcal{T}$  be symmetric. In other words, the stress-energy d-tensor of matter  $\mathcal{T}$  must satisfy the local symmetry conditions

$$\mathcal{T}_{AB} = \mathcal{T}_{BA}, \quad \forall A, B \in \left\{ 1, i, \begin{smallmatrix} (1) \\ (i) \end{smallmatrix} \right\}.$$

Moreover, we must "a priori" have the equality:  $\mathcal{T}_{(i)(j)}^{(1)(1)} h_{11} = g_{ij} \mathcal{T}_{11} h^{11}$ .

By direct computations, the geometrical Einstein equations (4.5) and (4.6) imply the following identities of the stress-energy d-tensor:<sup>2</sup>

$$\begin{aligned} \mathcal{T}_1^1 &\stackrel{\text{def}}{=} h^{11} \mathcal{T}_{11} = -\frac{R}{2\mathcal{K}}, & \mathcal{T}_1^m &\stackrel{\text{def}}{=} g^{mr} \mathcal{T}_{r1} = 0, & \mathcal{T}_{(1)1}^{(m)} &\stackrel{\text{def}}{=} h_{11} g^{mr} \mathcal{T}_{(r)1}^{(1)} = 0, \\ \mathcal{T}_i^1 &\stackrel{\text{def}}{=} h^{11} \mathcal{T}_{1i} = 0, & \mathcal{T}_i^m &\stackrel{\text{def}}{=} g^{mr} \mathcal{T}_{ri} = \frac{1}{\mathcal{K}} \left( g^{mr} R_{ri} - \frac{R}{2} \delta_i^m \right), \\ \mathcal{T}_{(i)}^{1(1)} &\stackrel{\text{def}}{=} h^{11} \mathcal{T}_{1(i)}^{(1)} = 0, & \mathcal{T}_{(1)(i)}^{(m)(1)} &\stackrel{\text{def}}{=} h_{11} g^{mr} \mathcal{T}_{(r)(i)}^{(1)(1)} = -\frac{R}{2\mathcal{K}} \delta_i^m, \\ \mathcal{T}_{(1)i}^{(m)} &\stackrel{\text{def}}{=} h_{11} g^{mr} \mathcal{T}_{(r)i}^{(1)} = 0, & \mathcal{T}_{(i)}^{m(1)} &\stackrel{\text{def}}{=} g^{mr} \mathcal{T}_{r(i)}^{(1)} = 0. \end{aligned} \quad (4.7)$$

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<sup>2</sup>Summing over both indices  $m, r$  is assumed in (4.9), over  $r$  in (4.7), and over  $m$  in (4.8) and (4.10).

Consequently, the following local identities for the stress-energy d-tensor of matter hold good:

$$\begin{cases} \mathcal{T}_{1/1}^1 + \mathcal{T}_{1|m}^m + \mathcal{T}_{(1)1}^{(m)}|_{(m)}^{(1)} = \mathcal{T}_{1/1}^1 = -\frac{1}{2\mathcal{K}} \frac{\delta R}{\delta t} \\ \mathcal{T}_{i/1}^1 + \mathcal{T}_{i|m}^m + \mathcal{T}_{(1)i}^{(m)}|_{(m)}^{(1)} = \mathcal{T}_{i|m}^m = \frac{1}{\mathcal{K}} \left( g^{mr} R_{ri} - \frac{R}{2} \delta_i^m \right)_{|m} \\ \mathcal{T}_{(i)/1}^{1(1)} + \mathcal{T}_{(i)|m}^{m(1)} + \mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} = \mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} = -\frac{1}{2\mathcal{K}} \frac{\partial R}{\partial y_1^i}, \end{cases} \quad (4.8)$$

where

$$\begin{aligned} \mathcal{T}_{1/1}^1 &\stackrel{def}{=} \frac{\delta \mathcal{T}_1^1}{\delta t} + \mathcal{T}_1^1 \varkappa_{11}^1 - \mathcal{T}_1^1 \varkappa_{11}^1 = \frac{\delta \mathcal{T}_1^1}{\delta t}, \quad \mathcal{T}_{1|m}^m \stackrel{def}{=} \frac{\delta \mathcal{T}_1^m}{\delta x^m} + \mathcal{T}_1^r L_{rm}^m = 0, \\ \mathcal{T}_{(1)1}^{(m)}|_{(m)}^{(1)} &\stackrel{def}{=} \frac{\partial \mathcal{T}_{(1)1}^{(m)}}{\partial y_1^m} = 0, \quad \mathcal{T}_{i/1}^1 \stackrel{def}{=} \frac{\delta \mathcal{T}_i^1}{\delta t} + \mathcal{T}_i^1 \varkappa_{11}^1 = 0, \\ \mathcal{T}_{i|m}^m &\stackrel{def}{=} \frac{\delta \mathcal{T}_i^m}{\delta x^m} + \mathcal{T}_i^r L_{rm}^m - \mathcal{T}_r^m L_{im}^r, \quad \mathcal{T}_{(1)i}^{(m)}|_{(m)}^{(1)} \stackrel{def}{=} \frac{\partial \mathcal{T}_{(1)i}^{(m)}}{\partial y_1^m} = 0, \\ \mathcal{T}_{(i)/1}^{1(1)} &\stackrel{def}{=} \frac{\delta \mathcal{T}_{(i)}^{1(1)}}{\delta t} + 2\mathcal{T}_{(i)}^{1(1)} \varkappa_{11}^1 = 0, \quad \mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} \stackrel{def}{=} \frac{\partial \mathcal{T}_{(1)(i)}^{(m)(1)}}{\partial y_1^m}, \\ \mathcal{T}_{(i)|m}^{m(1)} &\stackrel{def}{=} \frac{\delta \mathcal{T}_{(i)}^{m(1)}}{\delta x^m} + \mathcal{T}_{(i)}^{r(1)} L_{rm}^m - \mathcal{T}_{(r)}^{m(1)} L_{im}^r = 0. \end{aligned} \quad (4.9)$$

Taking into account that we have the  $y$ -independence  $R = R(x)$ , we obtain the following result:

**Corollary 4.4.** *The stress-energy d-tensor of matter  $\mathcal{T}$  must verify the following **conservation geometrical laws**:*

$$\mathcal{T}_{1/1}^1 = 0, \quad \mathcal{T}_{i|m}^m = \frac{1}{\mathcal{K}} \left[ g^{mr} R_{ri} - \frac{R}{2} \delta_i^m \right]_{|m}, \quad \mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} = 0. \quad (4.10)$$

## 4.2 Related electromagnetic model considerations

In the paper [13] an electromagnetic geometrical model was developed, based on a given Lagrangian function  $L(t, x, y)$  on the 1-jet space  $J^1(\mathbb{R}, M^n)$ . In the background of our electromagnetic geometrical formalism from [13], we work with an *electromagnetic distinguished 2-form*<sup>3</sup>

$$\mathbb{F} = F_{(i)j}^{(1)} \delta y_1^i \wedge dx^j,$$

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<sup>3</sup>We implicitly assume that the Latin letters run from 1 to  $n$ ; as well, we further denote by  $\mathcal{A}_{\{i,j\}}$  – the alternate sum, and by  $\sum_{\{i,j,k\}}$  – the cyclic sum.

where

$$F_{(i)j}^{(1)} = \frac{h^{11}}{2} \left[ g_{jr} N_{(1)i}^{(r)} - g_{ir} N_{(1)j}^{(r)} + (g_{ir} L_{jm}^r - g_{jr} L_{im}^r) y_1^m \right].$$

The electromagnetic components  $F_{(i)j}^{(1)}$  satisfy the following *Maxwell geometrical equations* [13]:

$$\begin{cases} F_{(i)j/1}^{(1)} &= \frac{1}{2} \mathcal{A}_{\{i,j\}} \left\{ \bar{D}_{(i)1|j}^{(1)} - D_{(i)m}^{(1)} G_{j1}^m + d_{(i)(m)}^{(1)(1)} R_{(1)1j}^{(m)} - \right. \\ &\quad \left. - \left( C_{j(m)}^{p(1)} R_{(1)1i}^{(m)} - G_{i1|j}^p \right) h^{11} g_{pq} y_1^q \right\}, \\ \sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} &= -\frac{1}{4} \sum_{\{i,j,k\}} \frac{\partial^3 L}{\partial y_1^i \partial y_1^p \partial y_1^m} \left( \frac{\delta N_{(1)j}^{(m)}}{\delta x^k} - \frac{\delta N_{(1)k}^{(m)}}{\delta x^j} \right) y_1^p, \\ \sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} \Big|_{(k)}^{(1)} &= 0, \end{cases}$$

where

$$\begin{aligned} \bar{D}_{(i)1}^{(1)} &= \frac{h^{11}}{2} \frac{\delta g_{im}}{\delta t} y_1^m, & D_{(i)j}^{(1)} &= h^{11} g_{ip} \left( -N_{(1)j}^{(p)} + L_{jm}^p y_1^m \right), \\ d_{(i)(j)}^{(1)(1)} &= h^{11} \left( g_{ij} + g_{ip} C_{m(j)}^{p(1)} y_1^m \right), & R_{(1)1j}^{(m)} &= \frac{\delta M_{(1)1}^{(m)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(m)}}{\delta t}, \\ \bar{D}_{(i)1|j}^{(1)} &= \frac{\delta \bar{D}_{(i)1}^{(1)}}{\delta x^j} - \bar{D}_{(m)1}^{(1)} L_{ij}^m, & G_{i1|j}^k &= \frac{\delta G_{i1}^k}{\delta x^j} + G_{i1}^m L_{mj}^k - G_{m1}^k L_{ij}^m, \end{aligned}$$

and we have

$$\begin{cases} F_{(i)j/1}^{(1)} = \frac{\delta F_{(i)j}^{(1)}}{\delta t} + F_{(i)j}^{(1)} \mathcal{A}_{11}^1 - F_{(m)j}^{(1)} G_{i1}^m - F_{(i)m}^{(1)} G_{j1}^m, \\ F_{(i)j|k}^{(1)} = \frac{\delta F_{(i)j}^{(1)}}{\delta x^k} - F_{(m)j}^{(1)} L_{ik}^m - F_{(i)m}^{(1)} L_{jk}^m, \\ F_{(i)j|k}^{(1)} \Big|_{(k)}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial y_1^k} - F_{(m)j}^{(1)} C_{i(k)}^{m(1)} - F_{(i)m}^{(1)} C_{j(k)}^{m(1)}. \end{cases}$$

**Example.** The Lagrangian function that governs the movement law of a particle of mass  $m \neq 0$  and electric charge  $e$ , which is displaced concomitantly into an environment endowed both with a gravitational field and an electromagnetic one, is given by

$$L(t, x^k, y_1^k) = mch^{11}(t) \varphi_{ij}(x^k) y_1^i y_1^j + \frac{2e}{m} A_{(i)}^{(1)}(t, x^k) y_1^i + \mathcal{F}(t, x^k), \quad (4.11)$$

where the semi-Riemannian metric  $\varphi_{ij}(x)$  represents the *gravitational potential* of the space of events  $M$ ,  $A_{(i)}^{(1)}(t, x)$  are the components of a d-tensor on the 1-jet space  $J^1(\mathbb{R}, M^n)$  representing the *electromagnetic potential* and  $\mathcal{F}(t, x)$  is a smooth *potential function* on the product manifold  $\mathbb{R} \times M$ . It is important to note that the jet Lagrangian function (4.11) is a natural extension of the Lagrangian (defined on the tangent bundle) used in electrodynamics by Miron and Anastasiei [10]. In our jet Lagrangian formalism applied to (4.11), the *electromagnetic components* are given by ([13])

$$F_{(i)j}^{(1)} = -\frac{e}{2m} \left( \frac{\partial A_{(i)}^{(1)}}{\partial x^j} - \frac{\partial A_{(j)}^{(1)}}{\partial x^i} \right),$$

and the second set of *Maxwell geometrical equations* reduce to the classical ones [13]:

$$\sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} = 0,$$

where

$$F_{(i)j|k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k} - F_{(m)j}^{(1)} \gamma_{ik}^m - F_{(i)m}^{(1)} \gamma_{jk}^m.$$

This fact suggests, in our opinion, applicative credibility for our extended electromagnetic geometrical model.

On our particular 1-jet space  $J^1(\mathbb{R}, M^4)$ , the jet conformal Minkowski metric (1.1) (via the Lagrangian function  $L = F^2$ ) produces the electromagnetic 2-form which due to (3.4) trivially vanishes

$$\mathbb{F} = 0.$$

In conclusion, the jet conformal Minkowski extended electromagnetic geometrical model constructed on the 1-jet space  $J^1(\mathbb{R}, M^4)$  is trivial. Namely, in our jet geometrical approach, the jet conformal Minkowski electromagnetism, produced only by the metric (1.1) alone, leads to null electromagnetic geometrical components and to tautological Maxwell-like equations. In our opinion, this fact suggests that the jet conformal geometrical structure (1.1) of the 1-jet space  $J^1(\mathbb{R}, M^4)$  is suitable for modeling gravitation rather than electromagnetism.

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